SI No of QP

: 5713

Unique Paper Code

: 42354401

Name of the Paper

: Real Analysis

Name of the Course

: B.Sc. (Prog) Physical Sciences/Mathematical Sciences

Semester

Duration: 3 Hours

Maximum Marks: 75

(Write your Roll No. on the top immediately on receipt of this question paper.) All questions are compulsory.

Attempt any two parts from each question.

- 1. (a) Define a countable set. Show that the set $\mathbb Q$ of rational numbers is countable.
 - (b) Define absolute value of a real number 'x'.

Find all $x \in \mathbf{R}$ that satisfy the following inequalities:

- (i) 4 < |x+2| + |x-1| < 5
- (ii) $|2x-1| \le x+1$

(c) (i) Define supremum of a non-empty bounded subset S of **R**.

- (ii) Show that a real number u is the supremum of a non-empty subset S of R if and only if it satisfies the following conditions:
- (1) $s \leq u$ for all $s \in S$.

(2) For each positive real number ε , there exists $s_{\varepsilon} \in S$ such that $u - \varepsilon < s_{\varepsilon}$.

- 2. (a) State the Archimedean Property of real numbers. Show that if $x \in \mathbb{R}$, then there exists a unique $n \in \mathbf{Z}$ such that $n-1 \le x < n$.
 - (b) Define the convergence of a sequence (x_n) of real numbers. Using the definition, evaluate the following limit

(ii)
$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

 $\lim_{n\to\infty} \left(\frac{(-1)^n n}{n^2+1}\right)$

(c) Let (x_n) be a sequence of real numbers that converges to x and suppose that $x_n \ge 0$, $\forall n \in \mathbb{N}$. Show that the sequence $(\sqrt{x_n})$ converges to \sqrt{x} .

(6,6)3. (a) Prove that every monotonically decreasing and bounded below sequence of real numbers converges.

(b) Show that the sequence (x_n) defined by $x_1 = 1$; $x_{n+1} = \frac{1}{4}(2x_n + 3)$, $\forall n \ge 1$ is convergent. Also, find $\lim_{n \to \infty} x_n$.

(c) State Cauchy's Convergence Criterion for sequences of real numbers. Show directly from the definition that the following sequence is a Cauchy sequence:

$$\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right).$$
 (6.5, 6.5)

4. (a) State and prove Comparison test for positive term series. Hence, show that the following series converges:

 $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$

- (b) Suppose that (x_n) is a sequence of non-negative real numbers. Prove that the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded.
- (c) (1) State (without proof) D'Alembert's ratio test for an infinite series.
 - (2) Test for convergence the series:
 - (i) $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + ---$
 - (ii) $\frac{1}{\log 2} + \frac{1}{(\log 3)^2} + \frac{1}{(\log 4)^3} + \cdots$

(6.5, 6.5)

- 5. (a) (i) Define an absolutely convergent series. Is every convergent series absolutely convergent? Justify your answer.
 - (ii) Test for convergence the series:
 - (ii) Lest for convergence the series: $(1) \ 1 \frac{1}{2} + \frac{1}{2^2} \frac{1}{2^3} + \frac{1}{2^4} \cdots$ $(2) \sum_{n=1}^{\infty} (-1)^n \cdot e^{-n}$ (b) Show that if a > 0, then the sequence $\left(\frac{nx}{1+n^2x^2}\right)$ converges uniformly on the interval $[a, \infty)$ but not uniform on the interval $[0, \infty)$.
 - (c) State Weierstrass West for uniform convergence of series. Hence, show that

$$\sum \frac{1}{x^2 + n^2}, \ \forall \ x \in \mathbf{R}$$

is uniformly convergent

(6.5,6.5)

6. (a) Find the radius of convergence and exact interval of convergence of the power series

$$\sum \frac{n+1}{(n+2)(n+3)} x^n.$$

- (b) Show that the function $f(x) = x^2$ defined on the interval [0, b], where b > 0 is Riemann
- (c) Show that every continuous function defined on [a, b] is Riemann integrable.

(6,6)